Spin Stability of a Satellite Equipped with Four Booms

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A discrete-element model is used to study the stability of simple spinning motions of a spacecraft consisting of a rigid body and four radial, elastic booms. The underlying theoretical basis is a method of stability analysis described previously. This work is intended to serve the two-fold purpose of showing how the method can be employed to deal with spacecraft containing elastic bodies and of giving specific results for a class of spacecraft of practical interest.

Introduction

SINCE the flight of Explorer I in 1958, spacecraft designers have become increasingly concerned with flexible body dynamics. In particular, much attention[1-13] has been focused on the effects of flexible appendages on the attitude stability of spacecraft.

One of the earliest studies of the effects of flexibility on spinning satellites was carried out by Buckens,[1] who examined equations of motion linearized about a state of constant spin for a satellite containing an arbitrary number (≥ 3) of radial rods and showed how the elastic modes couple with the rigid body modes. Hughes and Fung[2] and Kulla[3] also performed stability analyses for satellites containing an arbitrary number of radial rods. Hughes and Fung found that the "maximum axis rule" furnishes a necessary but not sufficient requirement for stability, and derived sufficiency criteria from a Liapunov function. Kulla studied the linearized system of ordinary and partial differential equations of spinning bodies containing elastic rods along and perpendicular to the spin axis and obtained stability criteria applicable to a satellite carrying rods along the spin axis and cable-like appendages perpendicular to the spin axis.

The particular case of the four-boom satellite was investigated by a number of authors. Etkin and Hughes[4,5] explained the rapid spin rate decay of Alouette I and Explorer XX in terms of the joint influence on bending of the antennas of solar heating and radiation pressure. Their theory was confirmed by flight test data. Vasilev and Kovtunenko[6] studied the stability of a spinning rigid body with four elastically mounted rigid radial rods. Meirovitch and Calico[7] presented two approaches to the stability analysis of a torque-free satellite consisting of a central rigid body and a number of elastic parts, one approach being modal analysis and the other the use of integral coordinates. The case of four radial and two axial booms was presented as an example. Etkin[8] showed that the coupling between rigid body and elastic modes of motion did not satisfactorily account for the spin decay of Alouette I. A linearized dynamic analysis for a spin stabilized spacecraft having four radial wire antennas with tip masses was performed by Longman and Fedor.[9] Flatley,[10] focusing on energy considerations, assumed that if the reference spin state is stable, then the total energy associated with that state must be less than the energy associated with a spin about an axis skewed to the reference spin axis. He then generated stability charts numerically and showed that an increase in boom stiffness has a stabilizing effect, whereas an increase in spin rate tends to destabilize. Dokuchaev[11] performed a stability analysis of a spinning rigid body with four elastic radial rods and one such axial rod. Using modal coordinates, he obtained analytical expressions for the boundaries of stable rotation regions. Vigneron[12] found that the linearized equations of motion uncouple into two sets, one describing flexural motion in the plane of spin, and another describing the nutational motions of the central rigid body and out-of-plane flexing of the antennas. A technique was outlined for determining the resonant frequencies and mode shapes from the linearized equations and it was concluded that sufficient conditions for stability are that both the central body and the total undeformed system satisfy the "maximum axis rule." Torque-free behavior, as described by nonlinear equations of motion of a four-boom satellite, was studied by Rakowski and Renard[13] by carrying out computer simulations, which showed that the modes of vibration of rotating appendages may differ significantly from those of nonrotating appendages for large ratios of spin rate to boom stiffness.

When the appendages on a spacecraft can be regarded as elastically mounted rigid bodies, the choice of a mathematical model to be employed for the purpose of stability analysis is a straightforward matter. Conversely, when the appendages are deformable bodies, many paths are open to the analyst. The first decision that then must be made is whether to use a continuum approach or to resort to discretization, that is, to a finite element analysis. But the necessity to make choices does not end there. In the first case one may work in various ways with the governing partial differential equations, or one may turn to a modal analysis. In the second case, one is faced with a plethora of possible finite element representations.

In the present work, a finite element approach is employed together with a method of stability analysis described previously[14] to study attitude stability of a spinning spacecraft equipped with four elastic booms. The method is described as follows: Let S be a deformable system having $N+6$ degrees of freedom in a Newtonian reference frame A, and let R be a reference frame in which the central principal axes of inertia of S are fixed. Then the configuration of S in A can be specified by six "external" coordinates governing the orientation of R in A and the position of the mass center of S in A, and N noncyclic "internal" coordinates can be used to specify the configuration of S in R. (The mass center of S is assumed to be at rest, or moving with a constant velocity, in A. Hence, of the six external coordinates, only the three governing the orientation of R in A are of interest.) In the absence of external forces, S is necessarily capable of moving in such a way that the angular velocity of R in A has a constant magnitude Ω and remains fixed in R, and hence in A, parallel to a central principal axis of inertia of S, while all of the internal coordinates remain equal to zero. Such a motion is called a "simple spin," and it is the stability of simple spins that is to be investigated. Once the internal coordinates of S have been selected, potential energy and dissipation functions are introduced, and a simple spin of interest is identified. A special set of orthogonal axes Z_1, Z_2, Z_3, originating at the mass center S^* of S, is chosen so as to ensure that the coor-

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dinates of a generic particle of S can be expressed as single-valued functions of the internal coordinates and that, moreover, when all coordinates vanish, Z_1, Z_2, Z_3 coincide with the central principal axes of inertia of S. The system moments of inertia about these axes then are formed, and certain partial derivatives of these quantities and of the potential energy are evaluated with respect to the internal coordinates. Next, a quadratic form is constructed, involving these derivatives and the system moments of inertia, for zero values of the internal coordinates. Finally, the quadratic form is tested for positive definiteness, which yields "internal" stability conditions, and a test is performed to verify that the motion satisfies the maximum axis rule for "external" stability.

Fundamental Equations

Certain equations from Ref. 14 will now be reproduced in order to render the subsequent analysis self-contained. Since this work will deal with the stability of spinning motions of S during which all internal coordinates $q_1, \cdots, q_N$ of S vanish (hereafter designated as the "reference state" of S) these equations will involve functions of $q_1, \cdots, q_N$, evaluated at $q_1 = \cdots = q_N = 0$. Such evaluation will be indicated by a tilde above the symbol representing the function in question. For example, if $f(q_1, \cdots, q_N)$ denotes a function of $q_1, \cdots, q_N$, then $\tilde{f} \triangleq f(0, \cdots, 0)$.

If z_1, z_2, z_3 form a dextral set of unit vectors respectively parallel to Z_1, Z_2, Z_3, then I_{jk} can be defined as $I_{jk} \triangleq z_j \cdot I \cdot z_k$ $(j,k=1,2,3)$, where I is the inertia dyadic of S for S^*. When S contains one or more rigid bodies, the contribution B_{jk} of a typical such body B to I_{jk} can be found after expressing B_{jk} in terms of the following quantities: B_1, B_2, B_3, the principal moments of inertia of B for the mass center B^* of B; C_{jk} $(j,k=1,2,3)$, defined as $C_{jk} \triangleq z_j \cdot b_k$ $(j,k=1,2,3)$, where b_k $(k=1,2,3)$ form a dextral set of unit vectors respectively parallel to principal axes of interia of B for B^*; and p_i $(i=1,2,3)$, defined as $p_i \triangleq p \cdot z_i$ $(i=1,2,3)$, where p is the position vector of B^* relative to S^*. If m_B is the mass of B, and if $\eta_{ijk\ell}$ is defined as $\eta_{ijk\ell} = \delta_{i\ell}\delta_{jk} - \delta_{ij}\delta_{\ell k}$ $(i,j,k,\ell=1,2,3)$, where δ_{ij} denotes the Kronecker delta, then

$$\tilde{B}_{jk} = B_i \tilde{C}_{ji} \tilde{C}_{ki} + m_B \tilde{p}_i \tilde{p}_\ell \eta_{ijk\ell} \quad (j,k=1,2,3) \tag{1}$$

Here, a summation convention is used; that is, a repeated index in a product indicates a summation in which this index taken on all values in its natural range.

Required partial derivatives of B_{jk} with respect to the internal coordinates may be obtained as follows: if a subscript comma followed by one or two letters signifies partial differentiations with respect to internal variables, for example, $B_{jk,r} \triangleq \partial B_{jk}/\partial q_r$ $(j,k=1,2,3;\ r=1,\cdots,N)$, then the first two derivatives of B_{jk} in the reference state are

$$\tilde{B}_{jk,r} = B_i(\tilde{C}_{ji,r}\tilde{C}_{ki} + \tilde{C}_{ji}\tilde{C}_{ki,r}) + m_B(\tilde{p}_{i,r}\tilde{p}_\ell + \tilde{p}_i\tilde{p}_{\ell,r})\eta_{ijk\ell} \quad (j,k=1,2,3;\ r=1,\cdots,N) \tag{2}$$

and

$$\tilde{B}_{jk,rs} = B_i(\tilde{C}_{ji,rs}\tilde{C}_{ki} + \tilde{C}_{ji,r}\tilde{C}_{ki,s} + \tilde{C}_{ji,s}\tilde{C}_{ki,r} + \tilde{C}_{ji}\tilde{C}_{ki,rs}) + m_B(\tilde{p}_{i,rs}\tilde{p}_\ell + \tilde{p}_{i,r}\tilde{p}_{\ell,s} + \tilde{p}_{i,s}\tilde{p}_{\ell,r} + \tilde{p}_i\tilde{p}_{\ell,rs})\eta_{ijk\ell} \quad (j,k=1,2,3;\ r,s=1,\cdots,N) \tag{3}$$

The external stability conditions or "maximum axis rule" for a simple spin about Z_1 in the reference state can now be written as

$$\tilde{I}_{11} - \tilde{I}_{22} > 0, \quad \tilde{I}_{11} - \tilde{I}_{33} > 0 \tag{4}$$

Furthermore, if V, a potential energy for S (a function of $q_1, \cdots, q_N$) is introduced, then the quadratic form $\tilde{Z}_{,rs}$ can be constructed from previously defined quantities. For a stress-free spin, that is, a motion such that $\tilde{V}_{,r} = 0$ $(r=1,\cdots,N)$ while Ω differs from zero, $\tilde{Z}_{,rs}$ is given by

$$\tilde{Z}_{,rs} = \tilde{V}_{,rs} + \Omega^2[(\tilde{I}_{22} - \tilde{I}_{11})^{-1}\tilde{I}_{12,r}\tilde{I}_{12,s} + (\tilde{I}_{33} - \tilde{I}_{11})^{-1}\tilde{I}_{13,r}\tilde{I}_{13,s} - 2^{-1}\tilde{I}_{11,rs}] \quad (r,s=1,\cdots,N) \tag{5}$$

and internal stability conditions are obtained by requiring that the matrix with $\tilde{Z}_{,rs}$ as its elements be positive definite. Equations (1-5) are the same as Eqs. (79-82), (87), respectively, of Ref. 14.

System Description

The spacecraft to be analyzed (see Fig. 1) consists of a rigid body B and four booms C,D,E,F, placed as follows: if X_1, X_2, X_3 are the principal axes of inertia of B for B^*, the mass center of B, then, when undeformed, C and D lie on opposite sides of B in the $X_1 - X_3$ plane and are parallel to and a distance a from X_3, while E and F lie on opposite sides of B in the $X_1 - X_2$ plane and are parallel to and a distance $\bar{a}$ from X_2. The attachment points of C and D are each a distance c from X_1, and those of E and F are each a distance $\bar{c}$ from X_1. C and D each have length L_1, and E and F each have length L_2.

In torque-free motion, the system can move so that the angular velocity of B in an astronomical reference frame A has a constant magnitude Ω and is parallel to X_1 (and hence remains fixed in A) while the booms remain normal to X_1. This motion will be recognized as a simple spin.

Inertia Properties

The central principal moments of inertia of B respectively associated with X_1, X_2, X_3 are B_1, B_2, B_3; B has mass M; C and D each have mass M_1; and E and F each have mass M_2.

The discrete-element modeling of the booms is accomplished as follows: each boom is divided into n massless beam elements of equal length, each element possessing the same principal planes of flexure and associated flexural rigidities as the boom itself. Axial deformations are neglected (but axial forces will be taken into account). The inertia properties of the booms are brought into evidence by placing at each node (endpoint of a beam element) a particle whose mass is half the sum of the masses of boom portions represented by the elements which meet at the node. Consequently, all interior nodes for booms C and D contain particles of mass M_1/n; the two end nodes contain particles of mass $M_1/2n$; and the corresponding particle masses for booms E and F are M_2/n and $M_2/2n$, respectively.

If lines are fixed in B in such a way that the the nodes for C,D,E,F lie on these lines when the booms are undeformed, then, during any deformation, the location of each particle relative to B can be described by two distances measured from the line on which the particle lies when the associated boom is undeformed. The four particles located at the attachment points of C,D,E,F to B cannot move relative to B, and the distances in question are thus equal to zero. Non-zero generalized coordinates $q_1, \cdots, q_{8n}$ are assigned to the remaining particles on the four booms as shown in Table 1, where odd subscripts refer to translations in principal planes of B parallel to X_1, and even subscripts refer to translations parallel to the $X_2 - X_3$ plane. Numbering is done in sequence, beginning at the attachment end of each boom, excluding the point of attachment.

In order to implement the aforementioned procedure, S is taken to be the system formed by B and C,D,E,F, and Z_1,Z_2,Z_3 are introduced as axes originating at the mass center S^* of S and parallel to X_1,X_2,X_3, respectively. Then, as required, Z_1,Z_2,Z_3 are central principal axes of S when $q_r=0$ $(r=1,\cdots,8n)$. In addition, it is helpful to define μ and f_1, f_2 as

$$\mu \triangleq M+2(M_1+M_2),\quad f_1 \triangleq -\mu^{-1}M_1n^{-1},$$

$$f_2 \triangleq -\mu^{-1}M_2n^{-1}$$

As shown in detail in Ref. 15, one can then use Eqs. (1-3) to form the following quantities for later substitution into the inequalities (4) and Eq. (5) (only nonzero values of $\tilde{I}_{12,r}, \tilde{I}_{13,r}, \tilde{I}_{11,rs}$ are shown):

$$\tilde{I}_{11}=B_1+2M_1\left[c^2+cL_1+L_1^2\left(\frac{1}{3}+\frac{1}{6n^2}\right)\right]$$
$$+2M_2\left[\bar{c}^2+\bar{c}L_2+L_2^2\left(\frac{1}{3}+\frac{1}{6n^2}\right)\right] \tag{6}$$

$$\tilde{I}_{22}=B_2+4M\mu^{-2}(M_1a+M_2\bar{a})^2$$
$$+2M_1[a-2\mu^{-1}(M_1a+M_2\bar{a})]^2$$
$$+2M_2[\bar{a}-2\mu^{-1}(M_1a+M_2\bar{a})]^2$$
$$+2M_1\left[c^2+cL_1+L_1^2\left(\frac{1}{3}+\frac{1}{6n^2}\right)\right] \tag{7}$$

$$\tilde{I}_{33}=B_3+4M\mu^{-2}(M_1a+M_2\bar{a})^2$$
$$+2M_1[a-2\mu^{-1}(M_1a+M_2\bar{a})]^2$$
$$+2M_2[\bar{a}-2\mu^{-1}(M_1a+M_2\bar{a})]^2$$
$$+2M_2\left[\bar{c}^2+\bar{c}L_2+L_2^2\left(\frac{1}{3}+\frac{1}{6n^2}\right)\right] \tag{8}$$

$$\tilde{I}_{12,2i}=\tilde{I}_{12,2n+2i}=2\tilde{I}_{12,2n}=2\tilde{I}_{12,4n}$$
$$=2Mf_1\mu^{-1}(M_1a+M_2\bar{a})-M_1[a-2\mu^{-1}(M_1a$$
$$+M_2\bar{a})](2f_1+n^{-1})-2M_2f_1[\bar{a}-2\mu^{-1}(M_1a+M_2\bar{a})] \tag{9}$$

$$\tilde{I}_{12,4n+2i-1}=-\tilde{I}_{12,6n+2i-1}=-M_2n^{-1}(\bar{c}+iL_2n^{-1}) \tag{10}$$

$$\tilde{I}_{12,6n-1}=-\tilde{I}_{12,8n-1}=-2^{-1}M_2n^{-1}(\bar{c}+L_2) \tag{11}$$

$$\tilde{I}_{13,4n+2i}=\tilde{I}_{13,6n+2i}=2\tilde{I}_{13,6n}=2\tilde{I}_{13,8n}$$
$$=2Mf_2\mu^{-1}(M_1a+M_2\bar{a})-M_2[\bar{a}-2\mu^{-1}(M_1a$$
$$+M_2\bar{a})](2f_2+n^{-1})-2M_1f_2[a-2\mu^{-1}(M_1a+M_2\bar{a})] \tag{12}$$

$$\tilde{I}_{13,2i-1}=-\tilde{I}_{13,2n+2i-1}=-M_1n^{-1}(c+iL_1n^{-1}) \tag{13}$$

$$\tilde{I}_{13,2n-1}=-\tilde{I}_{13,4n-1}=-2^{-1}M_1n^{-1}(c+L_1) \tag{14}$$

$$\tilde{I}_{11,2i2j}=\tilde{I}_{11,2n+2i\,2n+2j}=2Mf_1^2+2[2(M_1$$
$$+M_2)f_1^2+2M_1f_1n^{-1}+\delta_{ij}M_1n^{-1}] \tag{15}$$

$$\tilde{I}_{11,4n+2i\,4n+2j}=\tilde{I}_{11,6n+2i\,6n+2j}=2Mf_2^2+2[2(M_1$$
$$+M_2)f_2^2+2M_2f_2n^{-1}+\delta_{ij}M_2n^{-1}] \tag{16}$$

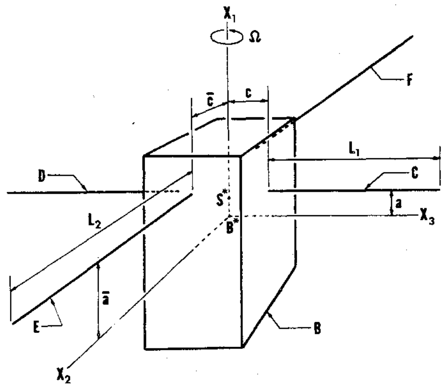

Fig. 1 Schematic representation of spacecraft.

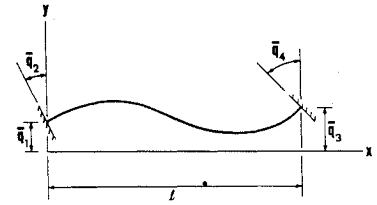

Fig. 2 General deformation of a beam.

$$\tilde{I}_{11,2i\,2n+2j}=2\tilde{I}_{11,2i\,2n}=2\tilde{I}_{11,2n+2i\,4n}=2\tilde{I}_{11,2i\,4n}$$
$$=2\tilde{I}_{11,2n+2i\,2n}=4\tilde{I}_{11,2n\,4n}=2Mf_1^2$$
$$+4[(M_1+M_2)f_1^2+M_1f_1n^{-1}] \tag{17}$$

$$\tilde{I}_{11,4n+2i\,6n+2j}=2\tilde{I}_{11,4n+2i\,6n}=2\tilde{I}_{11,6n+2i\,8n}=2\tilde{I}_{11,4n+2i\,8n}$$
$$=2\tilde{I}_{11,6n+2i\,6n}=4\tilde{I}_{11,6n\,8n}$$
$$=2Mf_2^2+4[(M_1+M_2)f_2^2+M_2f_2n^{-1}] \tag{18}$$

$$\tilde{I}_{11,2n\,2n}=\tilde{I}_{11,4n\,4n}=2^{-1}Mf_1^2$$
$$+(M_1+M_2)f_1^2+M_1n^{-1}(1+f_1) \tag{19}$$

$$\tilde{I}_{11,6n\,6n}=\tilde{I}_{11,8n\,8n}=2^{-1}Mf_2^2$$
$$+(M_1+M_2)f_2^2+M_2n^{-1}(1+f_2) \tag{20}$$

$$\tilde{I}_{11,rs}=\tilde{I}_{11,sr}\quad (r,s=1,\cdots,8n) \tag{21}$$

Elastic Properties

The principal planes of flexure of booms C,D,E,F are presumed to be parallel to central principal planes of inertia of body B. C and D have flexural rigidity EI_1 in both of their principal planes of flexure. Similarly, E and F have flexural rigidity EI_2.

To construct a potential energy function of precisely the generalized coordinates previously introduced for subsequent use in Eq. (5), the elastic properties of a single beam element of flexural rigidity EI and length ℓ will be studied first.

When a boom rotates about an axis perpendicular to its undeformed longitudinal axis in the manner of booms C,D,E,F, a centrifugal force acts on each element of the boom. These

Table 1 Generalized coordinates

Boom	Generalized coordinates
C	$q_1,\cdots,q_{2n}$
D	$q_{2n+1},\cdots,q_{4n}$
E	$q_{4n+1},\cdots,q_{6n}$
F	$q_{6n+1},\cdots,q_{8n}$

forces may be presumed to influence the transverse displacements of the boom while they, themselves, are not influenced significantly by these displacements; and one can account for them by letting an axial tensile force F_0 act on each end of the element, taking for F_0 the sum of the centrifugal forces associated with all particles of the discrete-element inertia model situated "outboard" of the element under consideration. The strain energy of the element is then[16]

$$V=\int_0^{\ell}\left\{\frac{EI}{2}[y''(x)]^2+\frac{F_0}{2}[y'(x)]^2\right\}dx \tag{22}$$

where $y(x)$ is the transverse displacement of the beam element at point x. To attribute displacements to a beam element, we consider the deflection associated with the four end displacements corresponding to the generalized coordinates $\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4$ shown in Fig. 2, which leads to[15]

$$y(x)=\left(1-\frac{3x^2}{\ell^2}+\frac{2x^3}{\ell^3}\right)\bar{q}_1+\left(x-\frac{2x^2}{\ell}+\frac{x^3}{\ell^2}\right)\bar{q}_2+\left(\frac{3x^2}{\ell^2}-\frac{2x^3}{\ell^3}\right)\bar{q}_3+\left(-\frac{x^2}{\ell}+\frac{x^3}{\ell^2}\right)\bar{q}_4 \tag{23}$$

Substitution into Eq. (22) yields

$$\begin{aligned}V=&\left(\frac{6EI}{\ell^3}+\frac{3F_0}{5\ell}\right)\bar{q}_1^2+\left(\frac{3EI}{\ell^2}+\frac{F_0}{20}\right)\bar{q}_1\bar{q}_2+\left(-\frac{6EI}{\ell^3}-\frac{3F_0}{5\ell}\right)\bar{q}_1\bar{q}_3\\&+\left(\frac{3EI}{\ell^2}+\frac{F_0}{20}\right)\bar{q}_1\bar{q}_4+\left(\frac{3EI}{\ell^2}+\frac{F_0}{20}\right)\bar{q}_2\bar{q}_1+\left(\frac{2EI}{\ell}+\frac{F_0\ell}{15}\right)\bar{q}_2^2\\&+\left(-\frac{3EI}{\ell^2}-\frac{F_0}{20}\right)\bar{q}_2\bar{q}_3+\left(\frac{EI}{\ell}-\frac{F_0\ell}{60}\right)\bar{q}_2\bar{q}_4\\&+\left(-\frac{6EI}{\ell^3}-\frac{3F_0}{5\ell}\right)\bar{q}_3\bar{q}_1+\left(-\frac{3EI}{\ell^2}-\frac{F_0}{20}\right)\bar{q}_3\bar{q}_2\\&+\left(\frac{6EI}{\ell^3}+\frac{3F_0}{5\ell}\right)\bar{q}_3^2+\left(-\frac{3EI}{\ell^2}-\frac{F_0}{20}\right)\bar{q}_3\bar{q}_4\\&+\left(\frac{3EI}{\ell^2}+\frac{F_0}{20}\right)\bar{q}_4\bar{q}_1+\left(\frac{EI}{\ell}-\frac{F_0\ell}{60}\right)\bar{q}_4\bar{q}_2\\&+\left(-\frac{3EI}{\ell^2}-\frac{F_0}{20}\right)\bar{q}_4\bar{q}_3+\left(\frac{2EI}{\ell}+\frac{F_0\ell}{15}\right)\bar{q}_4^2\end{aligned} \tag{24}$$

Equation (24) contains generalized coordinates corresponding to nodal rotations ($\bar{q}_2$ and $\bar{q}_4$), which have no counterparts in the inertia model. Hence, the attempt to use derivatives of V in Eq. (5) would give rise to incompatibilities. To resolve this dilemma, one can employ a technique applicable to stiffness matrices, for $V_{,rs}$, not V, is the quantity of interest, and it will now be shown that the matrix $[V_{,rs}]$ $(r,s=1,2,3,4)$ is, in fact, the stiffness matrix K of the beam element, that is, the matrix relating $\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4$ to corresponding generalized forces P_1,P_2,P_3,P_4, in accordance with the equation

$$P_r=k_{rs}\bar{q}_s \quad (r=1,2,3,4) \tag{25}$$

where k_{rs} $(r,s=1,2,3,4)$ are the elements of K. The proof proceeds as follows: The generalized force P_r associated with $\bar{q}_r$ can be expressed as

$$P_r=V_{,r}-D_r \quad (r=1,2,3,4) \tag{26}$$

where (see Ref. 14) D_r is a function of $\dot{\bar{q}}_r$ $(r=1,2,3,4)$ associated with energy dissipation, here attributed to structural damping, such that D_r vanishes when $\dot{\bar{q}}_1,\ldots,\dot{\bar{q}}_4$ all vanish, and (recall the summation convention) $D_r\dot{\bar{q}}_r\leq 0$, the equality sign holding only when $\dot{\bar{q}}_1,\cdots,\dot{\bar{q}}_4$ all vanish. (An example of a function which satisfies these requirements is $D_r=-c_{rs}\dot{\bar{q}}_s$ $(r=1,2,3,4)$, where c_{rs} $(r,s=1,2,3,4)$ are constants such that the matrix $[c_{rs}]$ is positive definite.) Upon differentiation, Eqs. (25) and (26) respectively yield $P_{r,s}=k_{rs}$ and $P_{r,s}=V_{,rs}$, from which it follows immediately that $V_{,rs}=k_{rs}$ $(r,s=1,2,3,4)$.

Differentiating Eq. (24) in accordance with this result, one can now express $[V_{,rs}]$ as the sum of two matrices, an elastic stiffness matrix K_E, given by

$$K_E=\begin{bmatrix}12EI/\ell^3 & 6EI/\ell^2 & -12EI/\ell^3 & 6EI/\ell^2\\ 6EI/\ell^2 & 4EI/\ell & -6EI/\ell^2 & 2EI/\ell\\ -12EI/\ell^3 & -6EI/\ell^2 & 12EI/\ell^3 & -6EI/\ell^2\\ 6EI/\ell^2 & 2EI/\ell & -6EI/\ell^2 & 4EI/\ell\end{bmatrix} \tag{27}$$

and what is referred to in the literature as a "geometric" stiffness matrix[17] K_G, given by

$$K_G=\begin{bmatrix}6F_0/5\ell & F_0/10 & -6F_0/5\ell & F_0/10\\ F_0/10 & 2F_0\ell/15 & -F_0/10 & -F_0\ell/30\\ -6F_0/5\ell & -F_0/10 & 6F_0/5\ell & -F_0/10\\ F_0/10 & -F_0\ell/30 & -F_0/10 & 2F_0\ell/15\end{bmatrix} \tag{28}$$

Now consider an element of boom C. The stiffness properties of this element for one of its principal planes of flexure are independent of (although the same as) those for the other principal plane of flexure. Therefore, a stiffness matrix of the form K_E+K_G can be constructed for each of the principal planes of flexure of the element. In all cases, $EI=EI_l$ and $\ell=L_l/n$, so that K_E is the same for all elements; however, K_G varies from element to element. For the nth element of boom C, $F_0=F_{0(n)}$, where $F_{0(n)}$, the axial force due to the acceleration of the tip mass of C, is given by:

$$F_{0(n)}=\frac{M_l}{2n}[c+L_l]\Omega^2 \tag{29}$$

The axial forces for the other elements of C can be generated in reverse order, recursively, by using the relation

$$F_{0(i-1)}=\frac{M_l}{n}[c+(i-1)\ell]\Omega^2+F_{0(i)} \quad (i=n,\cdots,2) \tag{30}$$

where again $\ell=L_l/n$. Once one has generated stiffness matrices corresponding to K_E+K_G for one principal plane of flexure of each element of C, one can construct two (identical) "overall" stiffness matrices for C, one for each principal plane of flexure, by using the "overlap" technique for element stiffness matrices. (A thorough account of the underlying theory is given in Ref. 18. As an example, the overall stiffness matrix for one principal plane of flexure of C, when $n=2$, is given in Eq. (36) of Ref. 15.) In similar fashion, an overall stiffness matrix can be generated for each principal plane of flexure of booms D,E,F.

Returning now to the question of incompatibility of the elastic and inertia discrete-element models, one can observe the following: The overall stiffness matrix for one principal plane of flexure of a boom is of order $2n+2$ and can be employed directly to relate $2n+2$ generalized coordinates to the same number of generalized forces. However, the lumped mass model only involves n degrees of freedom in each principal plane of flexure. To resolve this conflict, one may invoke the concept of stiffness matrix condensation. (A full treatment of this topic may be found in Ref. 19. See also Ref. 20. As an example of the condensation procedure, the two-element cantilever beam elastic stiffness matrix is condensed in Appendix A of Ref. 15 to remove nodal rotations.)

The overall stiffness matrix for each principal plane of flexure of each boom is condensed to remove nodal rotations. This leads to a new stiffness matrix, whose order is equal to the number of degrees of freedom in the lumped-mass model. In addition, these stiffnesses correspond precisely to the displacements of interest.

Referring once again to boom C, it can be seen that one of the two condensed overall stiffness matrices for C corresponds to the odd coordinates and the other to the even ones. By combining these two stiffness matrices, a single condensed stiffness matrix can be generated whose elements correspond to those and only those coordinates of boom C. Similarly, three other boom stiffness matrices can be generated, for booms D,E,F. Finally, a single stiffness matrix can be constructed for the entire spacecraft from the four boom stiffness matrices. This matrix is given by an equation of the following form

$$[V_{,rs}] = \begin{bmatrix} [K_C] & & & 0 \\ & [K_D] & & \\ & & [K_E] & \\ 0 & & & [K_F] \end{bmatrix} \quad (31)$$

where $[K_C],\cdots,[K_F]$ are respectively the condensed stiffness matrices for booms $C,\cdots,F$.

In summary, the procedure developed for forming $[V_{,rs}]$ can be implemented with a computer by proceeding as follows: 1) Use Eqs. (29) and (30) to generate axial forces for all of the elements of boom C. 2) Use Eq. (27), with $EI=EI_1$ and $\ell=L_1/n$ to form an elastic stiffness matrix for one beam element. 3) Use Eq. (28) to form one matrix with $\ell=L_1/n$ and $F_0=1$. 4) With the matrix obtained in Step 2, generate an elastic stiffness matrix for a principal plane of flexure of C, using the overlap technique. 5) Generate a geometric stiffness matrix for a principal plane of flexure of C, using the overlap technique, where the geometric stiffness matrix for each element is obtained by multiplying the matrix from Step 3 with the corresponding force from Step 1. 6) Form an overall stiffness matrix for one principal plane of flexure of C by adding the matrix from Step 4 to the one from Step 5. 7) Partition, rearrange, and condense the matrix from Step 6 to remove nodal rotations. 8) Generate the first submatrix of $[V_{,rs}]$ in Eq. (31) by making use of the fact that elements $(2i-1,2j-1)$ and $(2i,2j)$ of this matrix are the same as elements (i,j) of the matrix from Step 7, $(i,j=1,\cdots,n)$. The remaining elements of this submatrix are zeros. The second submatrix of $[V_{,rs}]$ in Eq. (31) is identical to the first. 9) Generate the other two submatrices of $[V_{,rs}]$ in Eq. (31) by repeating Steps (1-7) with $EI=EI_2$, $\ell=L_2/n$, and with M, c, L_1 respectively replaced with M_2, $\bar{c}$, L_2 in Eqs. (29) and (30), and then use a procedure similar to the one described in Step 8 to complete $[V_{,rs}]$.

Stability Conditions

The external stability conditions (4) can now be formulated by using $\bar{I}_{11}$, $\bar{I}_{22}$, $\bar{I}_{33}$, obtained, respectively, from Eqs. (6-8). In addition, the matrix $[\tilde{Z}_{,rs}]$ can be constructed in accordance with Eq. (5) by using $\bar{I}_{11}$, $\bar{I}_{22}$, $\bar{I}_{33}$, as well as $\bar{I}_{12,r}$, $\bar{I}_{13,r}$, $\bar{I}_{11,rs}$ from Eqs. (9-21), and $[\tilde{V}_{,rs}]$ (which is equal to $[V_{,rs}]$) as given in Eq. (31), and $[\tilde{Z}_{,rs}]$ can be tested for positive definiteness by using the method of Cholesky decomposition.[15]

Stability Charts

To illustrate the use of the method of stability analysis developed in preceding sections, we undertake a brief exploration of the relationship between physical parameters characterizing the spacecraft, on one hand, and stability, on the other hand. To this end we consider a vehicle formed by a central body with specified mass and central principal moments of inertia, and having booms attached at a definite distance from the nominal spin axis, but treat boom lengths, heights of boom attachment points above the central principal plane transverse to the spin axis, flexural rigidities, and nominal spin speed as variables. Specifically, we take $M=500$ kg, $B_1=110$ kg m^2 $B_2=100$ kg m^2, $B_3=70$ kg m^2, $c=\bar{c}=0.3$ m; set $a=\bar{a}$; make the booms identical by letting $L_2=L_1$, $I_2=I_1$; introduce a common mass density ρ for the booms, which makes it possible to express their masses as $M_1=M_2=\rho L_1=\rho L_2$; set $\rho=0.1$ kg/m, the value used on Alouette I[4]; let $EI_1=EI_2=145$ N m^2 (again from Alouette I[4]) and $\Omega=1$ rad/sec; repeatedly assign values to L_1 and a; for each pair of such values use Eqs. (6-21) and with $n=5$, implement the procedure for forming $[V_{,rs}]$; and finally, test for stability and display the results in the form of the stability chart‡ shown in Fig. 3. Next, we repeat the calculations after reducing the flexural rigidity of the booms by setting $EI_1=EI_2=1.45$ N m^2, which leads to Fig. 4, and then return to the earlier value of flexural rigidity, but increase the spin speed by making $\Omega=20$ rad/sec, to construct Fig. 5.

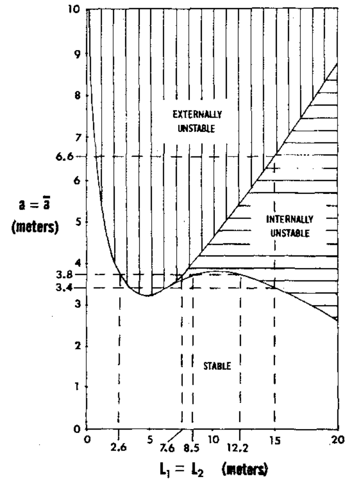

Fig. 3 Stability chart – $EI_1 = EI_2 = 145$ Nm^2, $\Omega = 1$ rad/sec.

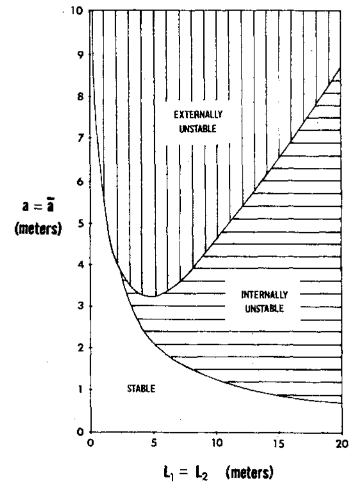

Fig. 4 Stability chart – $EI_1 = EI_2 = 1.45$ Nm^2, $\Omega = 1$ rad/sec.

Interpretation

Figures 3-5 contain information of practical significance. First, they show clearly that the so-called "maximum axis

‡The computations were performed also with $n=4$ and $n=6$. While the results for $n=4$ and $n=5$ differed from each other, those for $n=5$ and $n=6$ were the same. Hence, it suffices to take $n=5$.

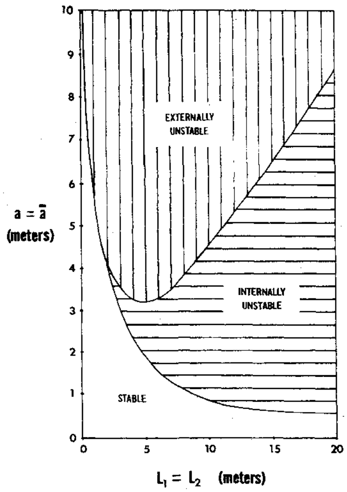

Fig. 5 Stability chart – $EI_1 = EI_2 = 145$ Nm^2, $\Omega = 20$ rad/sec.

rule'' tells but a portion of the stability story, for, while there appears in each chart a vertically shaded region, that is, a region representing failure to satisfy this rule, each chart contains, in addition, a region (indicated by horizontal shading) representing failure to satisfy further stability requirements, namely those arising from considerations of flexibility. This not only proves that flexibility can have an adverse effect on stability, but the charts permit one to explore such effects both quantitatively and qualitatively. Suppose, for example, that an attachment height $a = 3.8$ m has been chosen, but that the boom length remains to be assigned. Fig. 3 shows that inertia considerations, alone, would lead to the conclusion that one can take $L_1 < 2.6$ m or $L_1 > 7.6$ m, but that, in fact, the latter condition is inappropriately permissive, the correct range being $8.5 \text{ m} < L_1 < 12.2$ m. Similarly, if boom length is pre-assigned, say $L_1 = 15$ m, it appears that one is restricted to an attachment height such that $a < 3.4$ m, despite the fact that the maximum axis rule is now satisfied so long as $a < 6.6$ m.

Not surprisingly, a reduction in flexural rigidity tends to affect stability adversely, as may be seen by comparing Figs. 3 and 4, the latter containing a larger region of internal instability than does the former; and comparision of Fig. 5 with Figs. 3 and 4 reveals that an increase in spin speed can be essentially equivalent to a reduction in flexural rigidity. This conclusion was reached also by Flatley[10] and Rakowski and Renard.[13] To make detailed, quantitative comparisons is impracticable because of differences in mathematical models and parameter values employed by various authors.

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